Onthological models predictively inequivalent to quantum theory*

GianCarlo Ghirardi†
Department of Physics, University of Trieste, and the Abdus Salam ICTP, Trieste, Italy

Raffaele Romano‡
Department of Mathematics, Iowa State University, Ames, IA (USA)

Recently, it has been argued that no extension of quantum theory can have improved predictive power, under a strong assumption of free choice of the experimental settings, and validity of quantum mechanics. Here, under a different free choice assumption, we describe a model which violates this statement for almost all states of a bipartite two-level system, in a possibly experimentally testable way. From consistency with quantum mechanics and the non-signalling principle, we derive bounds on the local averages for the family of deterministic ontological theories our model belongs to.

PACS numbers: 03.65.Ta, 03.65.Ud
Keywords: Hidden variables, non-locality, free will

Introduction — In this work we address the following question: could a theory, which is conceived as a completion of quantum mechanics, be experimentally distinguishable from it? By *completion* we mean that the theory should be consistent with quantum mechanics, that is, it should fully reproduce all the quantum outcomes in a suitable regime, but it could provide a more refined description of the microscopic reality. By *experimentally distinguishable* we mean that there should not be physical principles making this deeper description fully unaccessible to any observer, irrespective of experimental complications. For instance, these are the features of classical mechanics when compared to statistical mechanics [19]. In practice, the completion involves the consideration of the so-called *ontic state* of the system, the (in principle) most accurate specification of its physical state. The request of the experimental testability of the completion does not require the precise knowledge of the ontic state, but only the accessibility of some information about it, which allows more accurate predictions than those implied by the mere knowledge of the quantum state vector.

Among others issues, the question we have raised at the beginning has been recently answered in the negative under a specific assumption of free measurement choice [2]. In this paper we exhibit a model which provides a positive answer, opening new perspectives on the debate of the completeness of quantum mechanics. For deterministic ontological models of this type describing a pair of two-level systems, we also derive constraints concerning how the local averages can differ from the quantum ones.

The family of completions of quantum mechanics has been usually denoted as *hidden variable models*, although more recently the term *ontological models* is preferred. In the past, the main motivation for their introduction was the attempt to provide a description of the micro-world as close as possible to that of the macro-world, interpreting all (or some of) the non-classical features of quantum mechanics (as probabilism and non-locality) as ignorance of the ontic state [1], supplying a richer information on the state of the system than that given by the state vector of standard quantum theory. After the theorems of Bell [3] and Kochen-Specker [4], proving that non-locality and contextuality are unavoidable in these theories, other issues are investigated, noticeably the meaning of the quantum state, as describing an element of reality or rather a state of knowledge [5–7], the possibility to deviate from quantum mechanics [8–11], the role of the measurement independence assumption [12, 13], and the dimension of the ontic state space [14–16]. We describe in more detail the aspects which are relevant for this work.

In quantum mechanics, the state of a system is represented by a vector ψ in a suitable Hilbert space. Observable quantities are represented by Hermitian operators, whose spectra contain the only possible measurement outcomes. For general states, these outcomes are known only probabilistically. Completeness of the theory corresponds to the assumption that ψ represents the most accurate information we can have concerning the real state of affairs, and that only a probabilistic knowledge of measurement outcomes is possible. We consider a bipartite system, the separated subsystems being identified by A and B. The observables pertaining to these subsystems are denoted by $A(a)$, $B(b)$, and the corresponding Hermitian operators by $\hat{A}(a)$, $\hat{B}(b)$, where a and b are vectors which identify the specific observables taken into account.

In an ontological model of quantum mechanics there is a deeper specification of the state of the system, the ontic state λ, living in a space which, for what concerns us, is not relevant to identify precisely. For sake of simplicity we assume that λ are continuous variables, but our considerations apply to completely general cases. The ontic state represents the complete description of the state of the system [20], which, however, is not fully accessible. To each state ψ is associated a distribution $\rho_\psi(\lambda)$, with $\rho_\psi(\lambda) \geqslant 0$ and

$$\int \rho_\psi(\lambda) d\lambda = 1 \qquad \text{for all}\, \psi. \tag{1}$$

These distributions might have overlapping supports for different ψ or not [1]. We limit our attention to deterministic ontological models, in which the measurement outcomes are fully specified by the ontic state (more generally, λ could determine only their probabilities). Contextuality implies that these outcomes can depend on the full context of measured observables; we will then denote them as $A_\psi(a,b,\ldots,\lambda)$ and $B_\psi(a,b,\ldots,\lambda)$, depending on the subsystem they refer to. Consistency of the ontological models with quantum mechanics means that all the quantum averages are reproduced, in particular

$$\int A_\psi(a,b,\ldots,\lambda)\rho_\psi(\lambda)d\lambda = \langle A(a)\rangle_\psi, \tag{2}$$

for the local averages, where $\langle A(a)\rangle_\psi = \langle\psi|\hat{A}(a)|\psi\rangle$ and similarly for B. But also quantum correlations should be obtained,

$$\int A_\psi(a,b,\ldots,\lambda)B_\psi(a,b,\ldots,\lambda)\rho_\psi(\lambda)d\lambda = \langle A(a)B(b)\rangle_\psi, \tag{3}$$

where $\langle A(a)B(b)\rangle_\psi = \langle\psi|\hat{A}(a)\otimes\hat{B}(b)|\psi\rangle$.

Notice that the ontic state λ is necessarily not fully accessible. In fact, in a theory like the one we are devising, it is just the ignorance of the precise value of λ which cancels non-locality in the averages (2), in accordance with the fact that quantum mechanics does not allow faster-than-light communication. Nonetheless, at least in principle, one can investigate whether it is possible to avoid superluminal communication while taking advantage of some information on λ. It is exactly this property which must characterize a completion of quantum theory which could be experimentally distinguishable from it, the crucial problem addressed in this paper.

With this in mind, suppose that λ is equivalently described by two variables (μ,τ), where μ denotes the unaccessible part, i.e. the one which must be averaged over, and τ the accessible one. Knowledge of τ should not allow superluminal communication; therefore, by writing $\rho_\psi(\lambda) = \rho_{\psi,\tau}(\mu)\rho_\psi(\tau)$, we can compute

$$\begin{aligned} \int A_\psi(a,b,\ldots,\lambda)\rho_{\psi,\tau}(\mu)d\mu &= f_\psi(a,\tau), \\ \int B_\psi(a,b,\ldots,\lambda)\rho_{\psi,\tau}(\mu)d\mu &= g_\psi(b,\tau), \end{aligned} \tag{4}$$

which are the local averages of the theory at the intermediate level. They express the *non-signalling conditions* in our context. It remains true that the theory reproduces quantum mechanics when τ is averaged over,

$$\begin{aligned} \int f_\psi(a,\tau)\rho_\psi(\tau)d\tau &= \langle A(a)\rangle_\psi, \\ \int g_\psi(b,\tau)\rho_\psi(\tau)d\tau &= \langle B(b)\rangle_\psi, \end{aligned} \tag{5}$$

but, if the state vector ψ is supplemented with the information on τ, the theory could be experimentally distinguishable from quantum mechanics [21]. Theories having this structure have been initially introduced by A. Leggett [8], and then further analyzed in the case of maximally entangled states [9, 10]. Here we provide one model fitting this class, describing arbitrarily entangled states.

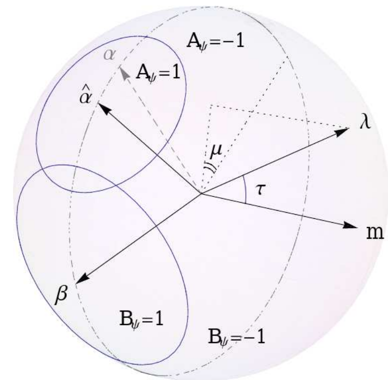

FIG. 1: Illustration of the crypto-nonlocal model with non-trivial local averages. The circles about $\hat{\alpha}$ and β are determined by the angles ξ and χ described in the text. The measurement settings lie on the plane orthogonal to the vector m, which identifies the direction of the incoming particle.

A model with non trivial local averages — The model is a generalization to arbitrary states of the famous Bell's model for the singlet state of a pair of two-level systems, and its details can be found elsewhere [18]. Here we review only the main results using a different notation, which is more convenient for the present purposes. An arbitrary state ψ is written as

$$|\psi\rangle = \gamma_0|00\rangle + \gamma_1|11\rangle, \tag{6}$$

where $\gamma_0^2+\gamma_1^2 = 1$, and $\gamma_0 \in [0, 1/\sqrt{2}]$ is a measure of entanglement: if $\gamma_0 = 0$, $|\psi\rangle$ is a separable state state; if $\gamma_0 = 1/\sqrt{2}$ it is a maximally entangled state. The hidden variable λ is a unit vector in the 3-dimensional real space and the pair $\{\psi,\lambda\}$ is identified with the ontic state. We consider local observables represented by the operators

$$\hat{A}(a) = \alpha_1 I + \alpha_2\sigma\cdot\alpha, \quad \hat{B}(b) = \beta_1 I + \beta_2\sigma\cdot\beta, \tag{7}$$

where $\alpha = (\alpha_x,\alpha_y,\alpha_z)$ and $\beta = (\beta_x,\beta_y,\beta_z)$ are real, unit vectors, $\alpha_1,\alpha_2,\beta_1,\beta_2 \in \mathbb{R}$ with $\alpha_2 \geqslant 0$, $\beta_2 \geqslant 0$, and $\sigma = (\sigma_x,\sigma_y,\sigma_z)$ is the vector of Pauli matrices. We can choose $a = (\alpha_1, \alpha_2\alpha_x, \alpha_2\alpha_y, \alpha_2\alpha_z)$, and similarly for b. For sake of simplicity, to develop our model we take $\alpha_1 = \beta_1 = 0$ and $\alpha_2 = \beta_2 = 1$, and assume that α and β lie in the plane orthogonal to the direction of propagation of the entangled particles [22]. The possible outcomes are in the set $\{-1, 1\}$, and they are defined as

$$A_\psi(a,b,\lambda) = \begin{cases} +1, & \text{if } \hat{\alpha}\cdot\lambda \geqslant \cos\xi, \\ -1, & \text{if } \hat{\alpha}\cdot\lambda < \cos\xi, \end{cases} \tag{8}$$

and

$$B_\psi(b,\lambda) = \begin{cases} +1, & \text{if } \beta\cdot\lambda \geqslant \cos\chi, \\ -1, & \text{if } \beta\cdot\lambda < \cos\chi. \end{cases} \tag{9}$$

In the previous relations, $\hat{\alpha} = \hat{\alpha}(\alpha, \beta)$ is in the plane of α and β, and given by $\hat{\alpha} \cdot \beta = \sin(\pi\alpha \cdot \beta/2)$ [23]; moreover,

$$\cos\xi = -\langle A(a)\rangle_\psi, \qquad \cos\chi = -\langle B(b)\rangle_\psi. \tag{10}$$

In [18] it has been proved that this model is predictively equivalent to quantum mechanics when λ is uniformly distributed on the unit sphere, $\rho(\lambda) = 1/4\pi$. At this point, we express λ by using polar coordinates (μ, τ), whose pole is identified by the direction of the incoming particle, see Fig. 1. They represent the unaccessible and the accessible part of λ, respectively. In (4) we have $\rho_{\psi,\tau}(\mu) = 1/2\pi$, and, by construction, integration over μ cancels non-locality in local averages. We find that

$$f_\psi(a,\tau) = \begin{cases} \frac{1}{\pi}\cos^{-1}\left(\frac{2\langle A(a)\rangle_\psi^2}{\sin^2\tau} - 1\right) - 1, & \text{if } |\tau - \frac{\pi}{2}| \leqslant \xi, \\ -1, & \text{if } |\tau - \frac{\pi}{2}| > \xi, \end{cases} \tag{11}$$

and a similar relation (with ξ replaced by χ) for $g_\psi(b,\tau)$, which clearly shows that in general $f_\psi(a,\tau)$ and $g_\psi(b,\tau)$ differ from the local quantum averages. While this toy-model is completely artificial, it provides evidence that models compatible with quantum mechanics, but in principle distinguishable from it, are indeed possible, with measurement settings a and b freely chosen.

On the local averages of any ontological theory — The requirements of equivalence with quantum mechanics and the non-signalling conditions put severe constraints on the probabilities at the intermediate level. In the following, we derive bounds on the local averages of any deterministic ontological model for quantum mechanics describing a pair of 2-level systems arbitrarily entangled, bounds which contain the single-outcome probabilities at the intermediate level. We adopt the aforementioned description of the ontic state in terms the statevector ψ and of an accessible part, τ, and a non-accessible one, μ, and assume that this splitting is independent of the local settings a and b.

While our derivation holds for both subsystems, for sake of simplicity we focus on the A part and its corresponding local average $f_\psi(a,\tau)$. As a measure of the distance between the local averages at the intermediate level of the ontological theory and quantum mechanics we introduce the quantity

$$\delta_\psi(a) = \int |f_\psi(a,\tau) - \langle A(a)\rangle_\psi|\, \rho_\psi(\tau) d\tau. \tag{12}$$

We consider a generic observable $A(a)$, represented in quantum mechanics by $\hat{A}(a)$ as in (7), with eigenvalues $\alpha_1 \pm \alpha_2$. For notational convenience we write $\hat{A}(\alpha) = \sigma \cdot \alpha$, and denote by $A(\alpha)$ the corresponding observable, and by $A_\psi(\alpha)$ its output. Since $[\hat{A}(a), \hat{A}(\alpha)] = 0$, the observables $A(a)$ and $A(\alpha)$ can be simultaneously measured, and then

$$A_\psi(a,b,\lambda) = \alpha_1 + \alpha_2 A_\psi(\alpha,b,\lambda). \tag{13}$$

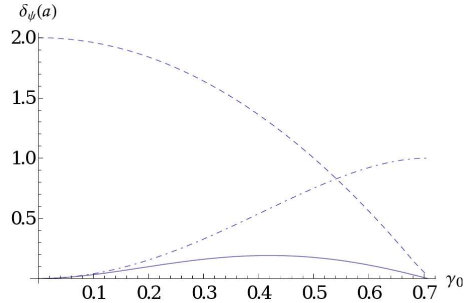

FIG. 2: Constraints for $\delta_\psi(a)$ as a function of entanglement, measured by γ_0. The dot-dashed line represent the constraint C1, the dashed line C2. The solid line represents $\delta_\psi(a)$ for the model described in the previous section. The observable taken into account is $\hat{A}(a) = \sigma_z$.

In fact, according to quantum mechanics, the outcomes of compatible observables must satisfy the same functional relations exhibited by the corresponding operators. Here, the setting b is arbitrary, and it is introduced to fully specify the measurement context. Notice that (a,b) and (α,b) specify the same context, i.e., the same complete set of commuting observables. By multiplying both terms of (13) by $\rho_{\psi,\tau}(\mu)$ and integrating over μ, we obtain

$$f_\psi(a,\tau) = \alpha_1 + \alpha_2 f_\psi(\alpha,\tau). \tag{14}$$

From (12), by using elementary properties of the absolute value under integration, we find a first constraint (denoted by C1 in the following),

$$\begin{aligned} \delta_\psi(a) &\leqslant \alpha_2 \int |A_\psi(\alpha,b) - \langle A(\alpha)\rangle_\psi| \rho_\psi(\lambda) d\lambda \\ &= \alpha_2\Big(1 - \langle A(\alpha)\rangle_\psi^2\Big), \end{aligned} \tag{15}$$

which however dos not rely on joint quantum correlations. We now derive a second constraint, which is based on them. From (12), we can write

$$\delta_\psi(a) \leqslant \alpha_2\Big(\int |f_\psi(\alpha,\tau)| \rho_\psi(\tau) d\tau + |\langle A(\alpha)\rangle_\psi|\Big), \tag{16}$$

and constraints on $f_\psi(a,\tau)$ can be derived by those on $f_\psi(\alpha,\tau)$. Since $\hat{A}(-\alpha) = -\hat{A}(\alpha)$, it follows that $A_\psi(-\alpha) = -A_\psi(\alpha)$, and then $f_\psi(-\alpha,\tau) = -f_\psi(\alpha,\tau)$.

Now consider $2n+1$ unit vectors γ_j, $j = 0, 1, \ldots, 2n$ such that $\gamma_0 = \alpha$ and $\gamma_{2n} = -\alpha$. Further assume that $2n$ pairs of local measurements are performed according to the following scheme: (i) when j is even, the local observables are given by $A(\gamma_j)$ and $B(\gamma_{j+1})$; (ii) when j is odd, the local observables are given by $A(\gamma_{j+1})$ and $B(\gamma_j)$. The measurement outcomes for $A(\gamma_j)$ and $B(\gamma_j)$ are ± 1. Therefore, by using that

$$\begin{aligned} &|A_\psi(\gamma_j,\gamma_{j+1},\lambda) - B_\psi(\gamma_j,\gamma_{j+1},\lambda)| = \\ &\quad 1 - A_\psi(\gamma_j,\gamma_{j+1},\lambda) B_\psi(\gamma_j,\gamma_{j+1},\lambda), \end{aligned} \tag{17}$$

after multiplication of both sides by $\rho_{\psi,\tau}(\mu)$ and integration over μ, it follows that

$$|f_\psi(\gamma_j,\tau) - g_\psi(\gamma_{j+1},\tau)| \leqslant 1 - E_{\psi,\tau}(\gamma_j,\gamma_{j+1}), \quad (18)$$

where we have defined the correlation at the intermediate level as

$$E_{\psi,\tau}(a,b) = \int A_\psi(a,b,\lambda) B_\psi(a,b,\lambda)\rho_{\psi,\tau}(\mu) d\mu \quad (19)$$

When j is odd, a similar argument leads to a relation analogous to (18), with γ_j and γ_{j+1} exchanged. By summing all these relations, and considering that $f_\psi(-\alpha,\tau) = -f_\psi(\alpha,\tau)$, we find that

$$|f_\psi(\alpha,\tau)| \leqslant n \; - \; \frac{1}{2}\sum_{k=0}^{n-1}\Big(E_{\psi,\tau}(\gamma_{2k},\gamma_{2k+1}) + \\ + E_{\psi,\tau}(\gamma_{2k+2},\gamma_{2k+1})\Big). \quad (20)$$

By further multiplying by $\rho_\psi(\tau)$ and integrating over τ,

$$\int |f_\psi(\alpha,\tau)|\rho_\psi(\tau) d\tau \leqslant \min_{\gamma_1,\ldots,\gamma_n n\in\mathbb{N}} \Omega_\psi(\alpha,n), \quad (21)$$

where we have defined

$$\Omega_\psi(\alpha,n) = n \; - \; \frac{1}{2}\sum_{k=0}^{n-1}\Big(\langle A(\gamma_{2k})B(\gamma_{2k+1})\rangle_\psi + \\ + \langle A(\gamma_{2k+2})B(\gamma_{2k+1})\rangle_\psi\Big), \quad (22)$$

and the minimum is taken by arbitrarily varying the vectors $\gamma_1,\ldots,\gamma_{2n-1}$, for any n. Therefore, by taking into account the joint correlations arising in an arbitrary number of measurements, we derive the second constraint (denoted by C2),

$$\delta_\psi(a) \leqslant \alpha_2\Big(\min_{n\in\mathbb{N}} \Omega_\psi(\alpha,n) + |\langle A(\alpha)\rangle_\psi|\Big). \quad (23)$$

Notice that, when ψ is a maximally entangled state, both terms in the r.h.s. of (23) vanish, and then $f_\psi(a,\tau) = \langle A(a)\rangle_\psi$ for all a (a rigorous proof, using analogous arguments, can be found in [11]). For arbitrary states, it is in general difficult to find the minimum in (23), therefore we have resorted to a numerical analysis. In general, the constraint C1 provides a better bound for poorly entangled states, whereas C2 is better when maximal entanglement is approached. In Fig. 2 we show how these constraints for $\delta_\psi(a)$ vary with entanglement.

Conclusions — In this work we have proven that ontological models of quantum theory which are compatible with it, but possibly experimentally distinguishable from it, are possible. In our model, the standard free will assumption, which involves the measurements of A and B, is satisfied, and superluminal communication is impossible. Our result seems to contradict a recent result implying that quantum mechanics is maximally informative [2], but this is not the case, since in our model we do not assume that the additional information on the ontic state must be acquired in a location and at a time which are spacelike separated with respect to A and B. Moreover, our model provides the first example of a crypto-nonlocal theory in which the local averages differ from the quantum mechanical ones for arbitrary non-maximally entangled states of a pair of qubits, and it is consistent with former results on maximally entangled states [9, 10, 17]. It proves that the local part of the hidden variables can be nontrivial, and suggests that possible deviations from quantum mechanics on the local averages could be observed also in the case of non-maximally entangled states.

Finally, we have derived explicit upper bounds on the local averages of any deterministic ontological theory for quantum mechanics, when the system is given by a pair of qubits. These constraints are determined by the requirements that: (i) the theory respects the non-signalling condition, when one takes into account the accessible part of λ, and (ii) it is compatible with quantum mechanics, that is, its predictions are the standard ones when the full average over λ is performed.

---

* This research is supported by the ARO MURI grant W911NF-11-1-0268.
† Electronic address: ghirardi@ictp.it
‡ Electronic address: rromano@iastate.edu

[1] N. Harrigan and R.W. Spekkens, Found. Phys. 40, 125 (2010)
[2] R. Colbeck and R. Renner, Nature Comm. 2, 411 (2011)
[3] J.S. Bell, Rev. Mod. Phys. 38, 447 (1966)
[4] S. Kochen and E.P. Specker, J. of Math. and Mech. 17, 59 (1967)
[5] M.F. Pusey, J. Barrett and T. Rudolph, Nature Phys. 8, 476 (2012)
[6] R. Colbeck and R. Renner, Phys. Rev. Lett. 108, 150402 (2012)
[7] P.G. Lewis, D. Jennings, J. Barrett and T. Rudolph, Phys. Rev. Lett. 109, 150404 (2012)
[8] A.J. Leggett, Found. Phys. 33, 1469 (2003)
[9] R. Colbeck and R. Renner, Phys. Rev. Lett. 101, 050403 (2008)
[10] C. Branciard, A. Ling, N. Gisin, C. Kurtsiefer, A. Lamas-Linares and V. Scarani, Phys. Rev. Lett. 99, 210407 (2007)
[11] G.C. Ghirardi and R. Romano, Phys. Rev. A 86, 022107 (2012)
[12] C. Brans, Int. J. Theor. Phys. 27, 219 (1988)
[13] M.J.W. Hall, Phys. Rev. Lett. 105, 250404 (2010)
[14] L. Hardy, Stud. Hist. Phil. Mod. Phys. 35, 267 (2004)
[15] A. Montina, Phys. Rev. A 83, 032107 (2011)
[16] A. Montina, Phys. Lett. A 375, 1385 (2011)
[17] G.C. Ghirardi and R. Romano, *About possible extensions of quantum theory*, in preparation
[18] G.C. Ghirardi and R. Romano, Phys. Rev. A 85, 042102 (2012)
[19] Obviously, in this case the deeper description is, in prin-

ciple, completely accessible.
[20] In this work, it will be apparent whether λ or rather $\{\psi, \lambda\}$ is considered as the ontic state of the system.
[21] We assume that τ could be prepared at the source, and then possibly communicated to A and B. But different scenarios are possible, for instance, the accessible and unaccessible parts of the ontic state could be contextual, that is, they could depend on a and b.
[22] Nonetheless, the model can be built in the fully general case, with completely analogous interpretation.
[23] Non-locality is apparent since the outcome of the measurement of A necessarily depends on b.